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LETTER TO THE EDITOR

Dirac quantisation of spin-2 field

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Abstract. We study the quantisation of the free (massless and massive) spin-2 field using Dirac's Hamiltonian method.

Although the spin-2 field is quite familiar (Pauli and Fierz 1939, Schwinger 1970), we have not seen an examination of the constraints and the Hamiltonian formalism, using Dirac's method (Dirac 1964). Our purpose in this letter is to give such an account.

Consider the following Lagrangian for a symmetric tensor field $\phi_{\mu\nu}(x)$,

$$L=\int \mathrm{d}^3x[\tfrac{1}{4}(\partial_\lambda\phi_{\mu\nu})^2-\tfrac{1}{2}(\partial_\mu\phi_{\mu\nu})^2+\tfrac{1}{2}\partial_\mu\phi_{\mu\nu}\partial_\nu\phi_{\lambda\lambda}-\tfrac{1}{4}(\partial_\mu\phi_{\nu\nu})^2-\tfrac{1}{2}M^2(\phi_{\mu\nu}\phi_{\mu\nu}-b\phi_{\mu\mu}\phi_{\nu\nu})]. \tag{1}$$

This Lagrangian corresponds in the massive case to the Pauli–Fierz form when b is set equal to unity (Salam and Strathdee 1976).

Decomposing the Lagrangian (1) into space and time, we obtain, up to total space and time derivatives,

$$\begin{aligned}L=\int \mathrm{d}^3x\{&\tfrac{1}{4}(\dot\phi_{ij})^2-\partial_i\phi_{0j}\dot\phi_{ij}+\partial_i\phi_{0i}\dot\phi_{jj}-\tfrac{1}{4}(\dot\phi_{ii})^2+\tfrac{1}{2}(\partial_i\phi_{0j})^2-\tfrac{1}{2}(\partial_i\phi_{i0})^2+\tfrac{1}{2}\partial_i\phi_{ij}\partial_j\phi_{00}\\&-\tfrac{1}{2}\partial_i\phi_{00}\partial_i\phi_{jj}-\tfrac{1}{4}(\partial_i\phi_{jk})^2+\tfrac{1}{2}(\partial_i\phi_{ij})^2-\tfrac{1}{2}\partial_i\phi_{ij}\partial_j\phi_{kk}+\tfrac{1}{4}(\partial_i\phi_{jj})^2\\&-\tfrac{1}{2}M^2[(\phi_{00})^2(1-b)-2(\phi_{0i})^2+(\phi_{ij})^2+2b\phi_{00}\phi_{ii}-b(\phi_{ii})^2]\}.\end{aligned} \tag{2}$$

Hence, we have the conjugate momenta

$$\begin{aligned}&\pi_{00}=0, \qquad \pi_{0i}=0,\\&\pi_{ij}=\tfrac{1}{2}\dot\phi_{ij}-\tfrac{1}{2}(\partial_i\phi_{0j}+\partial_j\phi_{0i})+\partial_k\phi_{0k}\delta_{ij}-\tfrac{1}{2}\dot\phi_{kk}\delta_{ij}.\end{aligned} \tag{3}$$

The Hamiltonian is

$$\begin{aligned}H=\int \mathrm{d}^3x\{&(\pi_{ij})^2-\tfrac{1}{2}(\pi_{kk})^2+2\pi_{ij}\partial_i\phi_{0j}-\tfrac{1}{2}\partial_i\phi_{ij}\partial_j\phi_{00}+\tfrac{1}{2}\partial_i\phi_{00}\partial_i\phi_{jj}\\&+\tfrac{1}{4}(\partial_i\phi_{jk})^2-\tfrac{1}{2}(\partial_i\phi_{ij})^2+\tfrac{1}{2}\partial_i\phi_{ij}\partial_j\phi_{kk}-\tfrac{1}{4}(\partial_i\phi_{jj})^2+\tfrac{1}{2}M^2[(\phi_{00})^2(1-b)\\&-2(\phi_{0i})^2+(\phi_{ij})^2+2b\phi_{00}\phi_{ii}-b(\phi_{ii})^2]\}.\end{aligned} \tag{4}$$

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The fundamental Poisson brackets are

$$\{\pi_{\mu\nu}(x), \phi_{\lambda\rho}(y)\} = \tfrac{1}{2}(\delta_{\mu\lambda}\delta_{\nu\rho} + \delta_{\mu\rho}\delta_{\nu\lambda})\delta^3(x-y). \tag{5}$$

From equations (3), we obtain the weakly vanishing (≈ 0) primary constraints

$$K_{00} \equiv \pi_{00} \approx 0, \qquad K_{0i} \equiv \pi_{0i} \approx 0. \tag{6}$$

Taking the Poisson brackets of $K_{0\mu}$ with H, we obtain the secondary constraints

$$\begin{aligned} C_0 &\equiv \partial_i\partial_j\phi_{ij} - \partial_i^2\phi_{jj} + 2M^2[\phi_{00}(1-b) + b\phi_{ii}] \approx 0 \\ C_i &\equiv \partial_j\pi_{ji} + M^2\phi_{0i} \approx 0. \end{aligned} \tag{7}$$

In the massless case ($M = 0$), the constraints K_{00}, K_{0i}, C_0 and C_i are all first class since they have vanishing Poisson brackets among each other. Hence, we choose corresponding to them, respectively, the following gauge fixing conditions,

$$\begin{aligned} \hat{K}_{00} &\equiv \phi_{00} \approx 0, \qquad \hat{K}_{0i} \equiv \phi_{0i} \approx 0, \\ \hat{C}_0 &\equiv \pi_{ii} \approx 0, \qquad \hat{C}_i \equiv \partial_j\phi_{ji} \approx 0. \end{aligned} \tag{8}$$

Thus we can eliminate from the twenty degrees of freedom $\phi_{\mu\nu}$ and $\pi_{\mu\nu}$, some sixteen corresponding to the totality of the constraints (6–8), leaving four degrees of freedom describing a massless spin-2 particle in phase space. The constraints can be put strongly equal to zero after defining modified Poisson (or Dirac) brackets (Dirac 1964) for the basic canonical variables ϕ_{ij} and π_{ij}. Hence from

$$\{C_0(x), \hat{C}_0(y)\} = -2\partial_i^2\delta^3(x-y), \tag{9}$$

we are led to the one-starred Dirac bracket

$$\begin{aligned} &\{\phi_{ij}(x), \pi_{kl}(y)\}^* \\ &\quad = \{\phi_{ij}(x), \pi_{kl}(y)\} - \int d^3z\{\phi_{ij}(x), \hat{C}_0(z)\}\frac{1}{2\partial^2}\{C_0(z), \pi_{kl}(y)\} \\ &\quad = \tfrac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\delta^3(x-y) - \tfrac{1}{2}\frac{1}{\partial^2}(\partial_k\partial_l - \partial^2\delta_{kl})\delta_{ij}\delta^3(x-y). \end{aligned} \tag{10}$$

Moreover, from

$$\{\hat{C}_i, C_j\}^* = -\tfrac{1}{2}(\partial^2\delta_{ij} + \partial_i\partial_j)\delta^3(x-y), \tag{11}$$

we obtain the two-starred Dirac brackets

$$\begin{aligned} &\{\phi_{ij}(x), \pi_{kl}(y)\}^{**} \\ &\quad = \{\phi_{ij}(x), \pi_{kl}(y)\}^* - \int d^3z\{\phi_{ij}(x), C_m(z)\}^* \\ &\qquad \times (2/\partial^2)[\delta_{mn} - \tfrac{1}{2}(\partial_m\partial_n/\partial^2)]\{\hat{C}_n(z), \pi_{kl}(y)\}^* \\ &\quad = \tfrac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\delta^3(x-y) - \tfrac{1}{2}\frac{1}{\partial^2}\delta_{ij}(\partial_k\partial_l - \partial^2\delta_{kl})\delta^3(x-y) \\ &\qquad + \tfrac{1}{2}\frac{1}{\partial^2}[\delta_{ik}\partial_j\partial_l + \delta_{il}\partial_j\partial_k + \delta_{jk}\partial_i\partial_l + \delta_{jl}\partial_i\partial_k - 2(\partial_i\partial_j\partial_k\partial_l/\partial^2)]\delta^3(x-y). \end{aligned} \tag{12}$$

Now we turn to the massive case ($M \neq 0$). If b is taken to be different from unity, all the constraints (6–7) would be second class and can eliminate eight degrees of freedom. The twelve degrees of freedom left are two more than is needed to describe a massive spin-2 particle in phase space. However, if b is set equal to unity, K_{00} is first class and $\phi_{00} \approx 0$ can be taken as a corresponding gauge fixing condition. Thus we eliminate ϕ_{00} and π_{00}. The constraints C_i and K_{0i} are second class and eliminate π_{0i} and ϕ_{0i}. This is done without the need to redefine the brackets of the remaining variables ϕ_{ij} and π_{ij}. The constraint C_0 is left alone as first class. Corresponding to it, we choose

$$\hat{C}_0 \equiv \pi_{kk} \approx 0 \tag{13}$$

as a gauge fixing condition. Now from

$$\{C_0(x), \hat{C}_0(y)\} = 2(-\partial^2 + 3M^2)\delta^3(x-y), \tag{14}$$

we are led to the Dirac brackets

$$\begin{aligned}
&\{\phi_{ij}(x), \pi_{kl}(y)\}^* \\
&\quad = \{\phi_{ij}(x), \pi_{kl}(y)\} + \int d^3z \{\phi_{ij}(x), \hat{C}_0(z)\}[1/2(-\partial^2 + 3M^2)]\{C_0(z), \pi_{kl}(y)\} \\
&\quad = \tfrac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\delta^3(x-y) + \tfrac{1}{2}\frac{1}{(-\partial^2 + 3M^2)}(\partial_k\partial_l - \partial^2\delta_{kl} + 2M^2\delta_{kl})\delta_{ij}\delta^3(x-y).
\end{aligned} \tag{15}$$

With the above results for the massless and the massive cases, one can make the straightforward transition to the canonical quantum theory (Dirac 1964) and to the path integral formulation (Faddeev 1970, Senjanovic 1976). Finally it is interesting to compare the above treatment of the spin-2 field with another high spin field of interest, namely the spin-$\frac{3}{2}$ field (Baaklini and Tuite 1978).

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